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THE SIMPLE MEANING OF COMPLEX RATES OF RETURN

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This note proposes a coherent system enabling interpretation and manipulation of rates of interest (or rates of return) including an imaginary component. This may help to shed new light on equations involving complex solutions, especially when valuing investment projects. In addition, a series of real rates can be associated with any complex rate. Each real rate can then be interpreted as a portfolio's expected return. As an example of application, when a project involves the joint production of two outputs whose markets have not the same risk, our approach allows the project's cash flow to be discounted at a single (but complex) rate.

INTRODUCTION

The economic and financial literature has long referred to the existence of rates of return including an imaginary component (see, for example, Dorfman (1981); Hazen (2003); Hartman and Schafrick (2004); Osborne (2005)), without, however, proposing an interpretation of these. Thus, as Osborne (2005) said: "The financial meaning of a complex solution that has an imaginary component is not clear. Perhaps there is none" (p. 165). In order to rectify this situation, this note proposes a coherent system enabling interpretation and manipulation of complex rates of interest. This may help to shed new light on equations involving complex solutions, especially when valuing investment projects. For instance, when a project involves the joint production of two outputs whose markets are subject to different risks, our approach allows the project's cash flows to be discounted at a single (but complex) rate.

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The interpretation of complex interest rates—or complex rates of return—is elaborated in the following section. We then show that a series of real rates can be associated with any complex rate. We then provide a practical application of our approach. The last section concludes.

A PROPOSED MEANING FOR COMPLEX RATES OF INTEREST

A Portfolio Composed of Two Assets with Differing Risks

If concrete meaning is to be given to complex internal rates of return, a symbolic meaning must be given to imaginary unit *i*. To do this, we make use of portfolios comprising two assets A and B. Because these assets are not subject to the same risk, they have different expected returns ρ_A and ρ_B for the period under consideration.

We assume here that it is possible to hold positive (i.e., long) or negative (i.e., short) positions for these two assets. The value of a portfolio is the sum of the value held in asset A and of that held in asset B. The risk of the portfolio, and therefore its expected return, depends on the proportion of each asset in the total value.

We now introduce the imaginary unit *i* as an exchange operator that leads to modification in the portfolio's composition and therefore in its risk.

Imaginary Unit i Interpreted As an Exchange Operator

Imaginary unit *i* will be used to symbolize the execution of two simultaneous operations: the sale of one dollar of asset A and the purchase of one dollar of asset B. These two operations in fact amount to exchanging one dollar of asset A for one dollar of asset B. For that reason, *i* is interpreted as an exchange operator that leads to a change in the risk but not the value of the portfolio.

Consequently, we note *iy* the transaction to exchange *y* dollars of asset A for *y* dollars of asset B. Because exchanging one dollar *y* times amounts to directly exchanging *y* dollars, we can write $yi = iy$. Such an exchange has a twofold impact on portfolio composition: the amount held in asset A is reduced by *y* dollars and the amount held in asset B is increased by *y* dollars.

We can also introduce an operator $-i$ to symbolize the reverse exchange operation, the exchange of one dollar of asset B for one dollar of asset A. Because $iy = (-i)(-y)$, there will be two possible interpretations of *iy*: the exchange of *y* dollars of asset A for *y* dollars of asset B or the exchange of $-y$ dollars of asset B for $-y$ dollars of asset A. These two operations in fact modify the portfolio in the same way, which in theory makes it possible to use both exchange operators in conjunction. However, for the sake of clarity, iy will in the rest of the article be interpreted as an operation exchanging y dollars of asset A for y dollars of asset B.

Consistently, the imaginary unit i will also be used to describe the composition of any given portfolio (i.e., the portfolio's amount invested in asset A and that invested in asset B). A portfolio will now be denoted $u + iv$, which reads as "u dollars invested in asset A plus v dollars invested in asset B." Here, the symbol i represents asset B and not an exchange operation as such. The portfolio's value is $u + v$.

Definition of a Complex Rate of Interest

We shall now suggest a meaning for the capitalization of a portfolio at the interest rate $a + ib$ over a period. To do this, we shall assume that this capitalization involves the investment of the two assets making up the portfolio (at their respective expected returns) over the period, followed, at the end of the period, by an exchange operation. These two operations are defined according to the following rules:

- During the period, the expected rate of return for asset A is $\rho_A = a + b$, and that for asset B is $\rho_B = a - b$; the return generated by each asset is immediately reinvested in the same asset.
- The exchange operation carried out at the end of the period involves an amount equal to b multiplied by the value of the portfolio at the start of the period.

As an illustration, we shall begin with consideration of a portfolio initially made up of a single asset (with a zero position in the other asset).

Let us first suppose that this portfolio comprises only one dollar of asset A (i.e., the portfolio is denoted as 1). The return (reinvested in asset A) expected at the end of the period stands at $a + b$ dollars. An exchange operation is then performed: b dollars of asset A are exchanged for b dollars of asset B. The portfolio available at the commencement of the following period therefore comprises $1 + a$ dollars of asset A and b dollars of asset B. This is consistent with the following calculation:

$$1 \times (1 + a + ib) = 1 + a + ib$$

Let us now suppose that the portfolio is made up of one dollar of asset B (i.e., the portfolio is denoted as i). The expected return at the end of the period is $a - b$ dollars, which is reinvested in asset B. Then b dollars of asset A are exchanged for b dollars of asset B, which generates a short position of b dollars in asset A. The resulting portfolio comprises $-b$ dollars of asset A and $1 + (a - b) + b = 1 + a$ dollars of asset B. This is consistent with the following calculation:

$$i \times (1 + a + ib) = -b + i(1 + a)$$

In more general terms, a portfolio is composed of the amount held in asset A and that held in asset B. The complex rate of interest should in this case be applied to each of these two amounts following the rules set out above, with the resulting positions held in each asset being then added together. As a result, portfolios and rates of interest can now be manipulated following the algebraic rules specific to complex numbers. Capitalization of the portfolio $u + iv$ at the complex rate $a + ib$, applying the rules previously defined, thus leads at the end of the period to a portfolio identical to the figure obtained if we multiply the two complex numbers:

$$(u + iv)(1 + a + ib) = (1 + a)u - vb + i((1 + a)v + bu)$$

We should note that in the context of discrete time we are using a (complex) capitalization factor written in its algebraic form. A demonstration is provided in the Appendix to show that the corresponding instantaneous rate of interest in continuous time can be expressed very simply as a function of the modulus and phase angle of this capitalization factor. The phase angle can be interpreted as the imaginary component of the instantaneous rate of interest.

Considering the imaginary unit i to be an exchange operator thus enables us to define, interpret, and manipulate complex rates of interest. Complex discounting can be defined immediately as the reverse operation to complex capitalization.

Special Cases

The capitalization of a portfolio at a real interest rate amounts to choosing assets A and B with the same expected return. It is as if the portfolio were composed exclusively of assets subject to the same risk. In this sense, our definition of a complex (discount or interest) rate generalizes the classic case.

Let us now look at one dollar of asset A capitalized at a factor i (equal to 1 plus the complex rate $-1 + i$). Because here $a + b = -1 + 1 = 0$, only one dollar, invested in asset A, is recovered at the end of the first period. This dollar of asset A is then exchanged for one dollar of asset B. This result shows clearly that the definition proposed for complex rates of interest is a generalization of the exchange operator (i.e., capitalizing at factor i does indeed amount to performing the corresponding exchange operation).

Extension of Complex Notation to Cash Flows

All cash flows received are susceptible to modify the existing portfolio. Consequently, we now extend to cash flows the complex notation initially defined for portfolios, by considering that the real component of a cash flow represents a (long or short) amount held in asset A and the imaginary component an amount held in asset B. Cash flows are therefore considered as bidimensional.

A complex cash flow is thus a portfolio in itself. In practice, for project valuation purposes, it may appear relevant to manipulate cash flows including an imaginary component; for example, when the project concerned generates two simultaneous cash flows with different risk exposures. The expected returns of assets A and B may then reflect these two distinct risk exposures. An example of this type is discussed later in the article.

Manipulation of (usual) cash flows defined as real numbers thus amounts to considering them as long positions in asset A when they are positive and short positions in asset A when they are negative. This is the implicit assumption in the classic calculation of a project's internal rate of return, where a negative cash flow, for instance, an investment outlay, can be viewed as a short position in capital (i.e., a firm's liability toward its capital providers). Using the project's real internal rate of return (i.e., with a zero imaginary component) allows for always considering a one-dimensional portfolio (i.e., with a position in asset B equal to zero). On the contrary, if the internal rate of return under consideration is the complex number $a + ib$, some of asset A is exchanged for some of asset B at the end of each period, with the expected returns on assets A and B being $a + b$ and $a - b$, respectively.

Interpretation of the Equation $i^2 = -1$

The algebra of complex numbers is entirely based on the postulated existence of the imaginary unit i whose sole extraordinary property is that its square is -1. For this reason, it can be seen to be fundamental to interpret this property in the context of our present approach.

Consider the problem raised by the calculation of the internal rate of return for a stream of two cash flows involving the spending of one dollar in period 0 followed by the spending of another dollar in period 2. The capitalization factor making this operation possible is either imaginary unit i or $-i$, because $-1 + i^2(-1) = -1 + (-i)^2(-1) = 0$. The internal rates of return of this stream of two cash flows are consequently $-1+i$ and $-1-i$.

Take the example of the complex rate $-1+i$. During the first period, capitalizing a negative real cash flow of one dollar (i.e., a short position of one dollar in asset A) at this rate does not generate any return, because asset A's expected return is here equal to zero $(= -1+1)$. Because the imaginary component of the rate is 1 and the portfolio's initial value is -1, a transaction consisting of the exchange of the short position of one dollar in asset A for a short position of one dollar in asset B is then performed at the end of the first period. Consequently, at the start of the second period, the portfolio consists in a short position of one dollar in asset B. During the second period, this portfolio is again invested at the rate $-1+i$, which corresponds to an asset B's expected return of -200% $(= -1-1)$. This therefore results in a long (positive) position of one dollar in asset B. Because the portfolio's value at the end of period 1 is still -1, a transaction consisting of the exchange of a short position of one dollar in asset A for a short position of one dollar in asset B is then performed again. This transaction offsets the long position of one dollar in asset B and generates a short position of one dollar in asset A. The upshot of this is therefore a short position of one dollar in asset A, corresponding to the spending of one dollar incurred in the second period.

Another interpretation can be given to the equation $-i^2 = 1$. We have seen previously that we could have introduced explicitly the operator $-i$, representing the exchange of one dollar of asset B for one dollar of asset A. The equation $-i^2 = i \times (-i) = 1$ can then be interpreted in the following way: successively applying two reverse exchange operations to a given portfolio takes us back to this portfolio.

WEIGHTED AVERAGE EXPECTED RETURNS ASSOCIATED WITH A COMPLEX RATE OF RETURN

By using the interpretations developed in the previous sections, we show here that a series of real rates of return can be associated with a complex rate of return. Each one of these real rates of return is the portfolio's return expected during a given period and is therefore a weighted average of expected returns ρ_A and ρ_B.

In brief, a portfolio's composition (i.e., the amounts invested in assets A and B respectively) is now described as a complex number. The risk of the portfolio, and therefore its expected return, depends on this composition. Its value, which corresponds to the total amount that would result from its liquidation, is a real number equal to the sum of the real and imaginary components of the complex number describing its composition. In the same way, a cash flow is described by a complex number. The approach used in this section is analogous to that enabling determination of the discount rate to be used with the standard weighted average cost of capital (WACC) method in classic capital budgeting. For instance, Chambers et al. (1982) and Pierru (2009) used it to demonstrate the consistency of various conventional methods.

The following notations relate to a cash flow stream covering N periods. The expected cash flow for year n ($n = 0, 1, \ldots, N$) is $u_n + iv_n$. Its expected total amount is therefore $u_n + v_n$, where u_n is invested in asset A and v_n in asset B. It is assumed that this cash flow stream is to be discounted at the complex rate $a + ib$.

The value of the cash flow stream under consideration corresponds to the value of a portfolio composed of assets A and B, with $\rho_A = a + b$ and $\rho_B = a - b$. Each year, the composition of this portfolio is given by the discounted sum of the subsequent cash flows. Let us denote this composition at the end of year n as $U_n + iV_n$. We have:

$$U_n + iV_n = \sum_{k=n+1}^{N} \frac{u_k + iv_k}{(1 + a + ib)^{k-n}} \qquad n = 0, 1, \ldots, N - 1 \tag{1}$$

$$U_N + iV_N = 0 \tag{2}$$

Equation (2) simply means that positions in both assets A and B are equal to zero at the end of year N (when all cash flows have already been generated).

Equation (1) can also be rewritten using recurrence:

$$U_n + iV_n = \frac{U_{n+1} + u_{n+1} + i(V_{n+1} + v_{n+1})}{1 + a + ib}$$

Let α_n be the proportion of the value of the portfolio held in asset A at the end of year n:

$$\alpha_n = \frac{U_n}{U_n + V_n} \tag{3}$$

The (average) return w_n on the portfolio expected in year $n + 1$ is therefore:

$$w_n = \alpha_n \rho_A + (1 - \alpha_n)\rho_B \tag{4}$$

The real discount rate w_n, defined as a weighted average expected return (WAER) in Equation (4), must be applied to the total amounts of the cash flows from year $n + 1$. In year 0, the value of the portfolio $U_0 + V_0$ is therefore given by the following relationship:

$$U_0 + V_0 = \sum_{n=1}^{N} \frac{u_n + v_n}{\prod_{k=0}^{n-1} (1 + w_k)} \tag{5}$$

For a given stream of cash flows, it is thus possible in this way to associate a series of real discount rates (i.e., a series of WAER) with a complex discount rate. The value of the portfolio can be obtained:

- either by discounting the cash flows at the complex rate
- or by discounting the total cash flow amounts with the associated series of real rates.

It should be noted that discounting all the cash flows at the same complex rate leads to variations in portfolio composition (i.e., α_n is not constant). As a consequence, a complex discount rate corresponds to a series of real discount rates (and not a single real rate).

As a numerical illustration, consider the complex rate of interest $0.1 + 0.08i$. We therefore have $\rho_A = 0.18$ and $\rho_B = 0.02$. Let one dollar be invested in year 0 in asset A and capitalized for 3 years at that rate. The value of the portfolio is \$1.60 by the end of year 3, since we have:

$$(1.1 + 0.08i)^3 = 1.31 + 0.29i$$

This value can also be determined with the (WAER) real rates. By using (1), (3), and (4), we obtain:

$$\begin{aligned} U_0 + iv_0 &= 1, U_1 + iv_1 = 1.1 + 0.08i, U_2 + iv_2 = (1.1 + 0.08i)^2 \\ &= 1.2036 + 0.176i \\ \alpha_0 &= 1, \alpha_1 = \frac{1.1}{1.18} = 0.93, \alpha_2 = \frac{1.2036}{1.2036 + 0.176} = 0.87, \\ w_0 &= 0.18, w_1 = (0.93 \times 0.18) + (0.07 \times 0.02) = 0.169 \\ w_2 &= (0.87 \times 0.18) + (0.13 \times 0.02) = 0.16. \end{aligned}$$

We thus have:

$$(1 + w_0)(1 + w_1)(1 + w_2) = 1.18 \times 1.169 \times 1.16 = 1.60.$$

We can now apply this result to a complex internal rate of return, denoted $a + ib$, of a classic stream of real cash flows. We need only choose $a + ib$

such that, in year 0, the value of the portfolio is the opposite of the cash flow. Therefore, retaining the previous notations, we must have:

$$U_0 + u_0 = \sum_{n=0}^{N} \frac{u_n}{(1 + a + ib)^n} = 0 \tag{6}$$

Equation (6) is a specific case of (1) since here all cash flows are real. In short, (6) simply defines $a + ib$ as being an internal rate of return of the investment project generating the stream of cash flows $\{u_0, u_1, \ldots, u_N\}$. By using (3) and (4), a series of real rates w_k can therefore be associated with $a + ib$, and (5) is here written:

$$u_0 + \sum_{n=1}^{N} \frac{u_n}{\prod_{k=0}^{n-1} (1 + w_k)} = 0$$

As shown previously, the real rate w_k can be interpreted as the portfolio's return expected for period $k + 1$. A series of WAER can therefore be associated with each project's complex rate of return. When the project's internal rate of return considered, denoted as r, is a real number, the series of WAER coincides with this rate (i.e., for every k: $w_k = \rho_A = r$).

EXAMPLE OF APPLICATION: INVESTMENT PROJECT WITH JOINT PRODUCTS WHOSE MARKETS HAVE DIFFERENT RISKS

Consider a company planning to invest \$15 million in industrial equipment enabling production of good A for the following 3 years. Due to the nature of the envisaged industrial process, a co-product known as good B is produced simultaneously. Because these goods A and B are sold on distinct markets, they generate two cash flow streams whose exposures to risk are different. The expected annual (income) cash flow is \$6 million for good A and \$500,000 for good B. In the absence of further information, is it possible to determine an internal rate of return for this investment project? The conventional approach does not provide us with any straightforward solution to this problem, because cash flows with different exposures to risk[1] should be discounted at different rates. Because the risks of the two cash flow streams differ, it does not seem possible to take them together to calculate a single internal rate of return for the project as a whole.

[1] **According to the Capital Asset Pricing Model (CAPM), only the exposure of an asset's return to the systematic risk involves a risk premium in the asset's expected return.**

We now propose to apply our approach in order to determine and interpret a complex internal rate of return. To do this, we shall consider that the project's annual cash flow is bidimensional and equal to $6 + 0.5i$ (in million dollars). According to the interpretation elaborated in previous sections, this (complex) notation allows us to account for the different risk exposures of the two cash flow streams. Consequently, we will determine the expected returns on assets A and B, considering that asset A (asset B) is as risky as the cash flows generated by the sale of good A (good B). Because the aim of the project is above all to make product A, the investment made in year 0 is here assumed to be fully invested in asset A. Every internal rate of return for the project, denoted $a + bi$, satisfies the following equation:

$$-15 + \sum_{k=1}^{3} \frac{6 + 0.5i}{(1 + a + bi)^k} = 0$$

For our numerical illustration, of the three corresponding internal rates of return, we shall consider that equal to $0.098 + 0.047i$. We therefore have $\rho_A = a + b = 14.5\%$ and $\rho_B = a - b = 5\%$.

The project can in this way be considered as a main investment generating an expected return of 14.5% in relation to good A. The cash flows generated by good B offer an expected return of 5%.

In order to enrich the analysis, we can now reverse the problem by assuming that the expected cash flows from goods A and B must be discounted at the respective rates of 14.5 and 5%. One first possibility would be to perform a net present value calculation based on the complex rate $0.098 + 0.047i$ obtained by considering that 14.5 and 5% reflect the expected returns on assets A and B, respectively. In accordance with our previous calculations, the net present value obtained is, of course, zero. The next possibility is to apply the classic valuation process in corporate finance, which is to discount each cash flow at a rate reflecting its risk. The net present value of the project then becomes:

$$-15 + \sum_{k=1}^{3} \frac{6}{(1 + \rho_A)^k} + \sum_{k=1}^{3} \frac{0.5}{(1 + \rho_B)^k}$$
$$= -15 + \sum_{k=1}^{3} \frac{6}{(1.145)^k} + \sum_{k=1}^{3} \frac{0.5}{(1.05)^k} = 0.18$$

Although small, the net present value is not zero. The discrepancy observed here comes from the fact that the two calculations are based on quite different assumptions. In the classic approach, it is implicit that the flows generated are reinvested in an activity presenting the same risk. In our approach, it is assumed that the flows generated are partly invested in an activity with a different risk. The advantage of our approach lies in the fact that it enables cash flows with different risks to be discounted at a single (but complex) rate.

Furthermore, by using (1), (3), and (4), we can determine the series of WAER associated with the complex rate of return $0.098 + 0.047i$. Let us begin with the discount rate to be applied to the value of the cash flows in year 3:

$$U_2 + iv_2 = \frac{6 + 0.5i}{1.098 + 0.047i} = 5.474 + 0.221i$$

$$\alpha_2 = \frac{5.474}{5.474 + 0.221} = 0.961$$

$$w_2 = (0.961 \times 0.145) + ((1 - 0.961) \times 0.05) = 0.141$$

Proceeding in like fashion, we obtain $w_1 = 0.143$ and $w_0 = 0.145$. Equation (5) becomes:

$$-15 + \frac{6.5}{1.145} + \frac{6.5}{1.145 \times 1.143} + \frac{6.5}{1.145 \times 1.143 \times 1.141} = 0$$

CONCLUSION

This article shows that complex rates of return have meaning. To provide an interpretation, we show how a portfolio made up of amounts invested in two distinct assets A and B can be capitalized at a complex rate of interest. This rate defines the return on each asset over the period considered and the amount of asset A exchanged for asset B at the end of the period. In order to introduce our approach we considered that the two assets were subject to different risks and therefore had different expected returns. As a result, a series of real rates of return can be associated with a complex rate of return. Each one of these real rates is the portfolio's return expected during a given period and is therefore a weighted average of the returns expected on assets A and B. We are aware of the apparently limited practical interest of the interpretations proposed in this article. However, as Osborne (2005) emphasized, "Many areas of finance remain unexamined in the light of the complex solutions to the equation" (p. 171). Giving meaning to those solutions can therefore throw new light on the problem under consideration. In addition, by considering joint products whose markets have different risks, we show that manipulating complex cash flows may have practical interest.

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APPENDIX

Let ρ be the modulus and θ ($\in]-\pi, \pi]$) the phase angle of the capitalization factor $1 + a + ib$, where:

$$1 + a + ib = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta} = e^{\ln\rho + i\theta}$$

This formulation suggests a straightforward interpretation of the modulus and phase angle of a complex capitalization factor. For this we shall define an "instantaneous" complex rate of interest to make it possible to capitalize and discount a portfolio in continuous time. This definition generalizes the classic case of a real rate of interest in continuous time.

We now introduce directly the instantaneous rate of interest $\ln\rho + i\theta$. By definition, investing in t a portfolio $f(t)$ (formed of the amounts invested in assets A and B) at this rate for a period dt leads to a portfolio $f(t + dt)$, where:

$$f(t + dt) = (1 + (\ln\rho + i\theta)dt)\, f(t) \qquad \text{(A1)}$$

By rearranging Equation (A1) we obtain:

$$\frac{df(t)}{dt} = (\ln\rho + i\theta)\, f(t) \qquad \text{(A2)}$$

Equation (A2) is a differential equation on f whose solution takes the form:

$$f(t) = f(0)e^{(\ln\rho + i\theta)t} = f(0)\rho^t e^{i\theta t} = f(0)(1 + a + ib)^t \qquad \text{(A3)}$$

Equation (A3) proves that the instantaneous rate of interest $\ln\rho + i\theta$ defined in continuous time is equivalent to the rate of interest $a + ib$ defined in discrete time.

BIOGRAPHICAL SKETCH

Axel Pierru is a senior economist at the IFP (France). He has research interests in capital budgeting, operational research, and energy economics.