Zombie outbreak with the '28 Days Later' effect

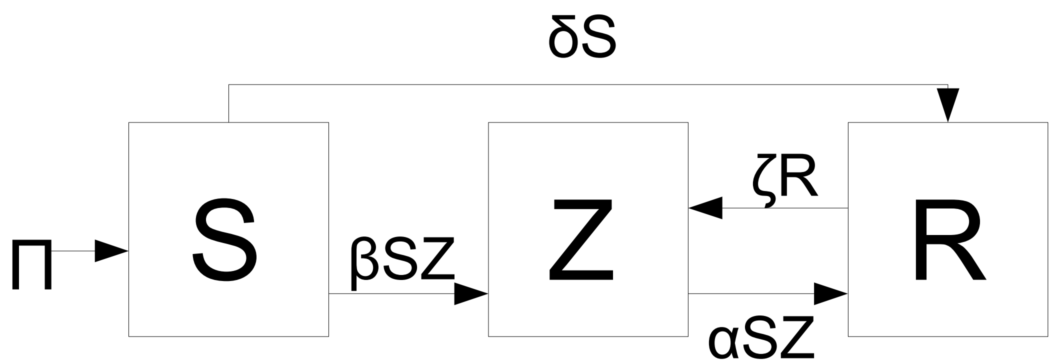
Tobi Lehman
Computational Physics Final Project

Abstract

We investigate a variant of the SZR model[1] with a time dependent term to simulate the type of zombie infestation depicted in the movie '28 Days Later'. In the movie, the zombies would only last about a month before dying. We predict that this time dependence will significantly change the dynamics, potentially even allowing for equilibria between nonzero values of the susceptible (S) and zombie (Z) populations.

The Basic Model and it's Modification

The model used in Munz et al[1] is based off of an SIR model that models real infectious diseases, the difference is that the SZR model allows for the removed (dead) population to contribute to the zombie (undead) population, the model can be visualized as follows:



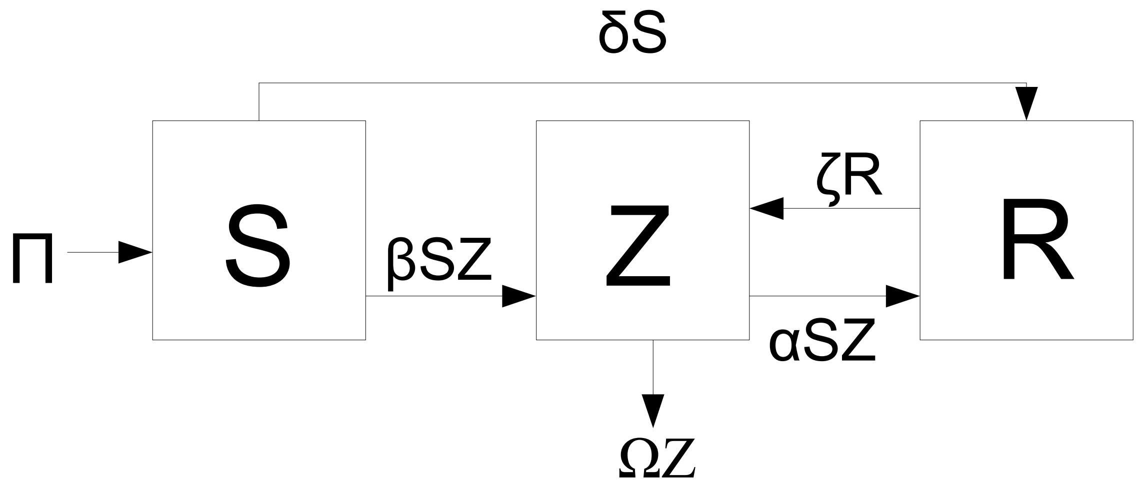
Where Π is the constant birth rate, δ is the 'natural' death parameter, β is the transmission parameter, ζ is the zombie parameter and α is the defeated zombie parameter. The differential equations based on this model are:

$$\frac{dS}{dt} = \Pi - \beta SZ - \delta S$$
$$\frac{dZ}{dt} = \beta SZ + \zeta R - \alpha SZ$$
$$\frac{dR}{dt} = \delta S + \alpha SZ - \zeta R$$

In [1], the above equations are simplified by investigating short time scales, where the birth and natural death rates are unimportant, thus setting $\Pi = \delta = 0$. In this paper, long term effects are explored (so nonzero values of Π and δ are assumed). Also, the model is modified to have a parameter for Z that is analogous to δ, this is the '28 Days Later' parameter, which we call Ω.

The Modified Model

We assume that that rate at which the zombies expire is proportional to the zombie population value Z, so the modified model is:



Note that the ΩZ term does not flow into the removed (dead) population, because that would feed back into the zombie population. The point of this term is to model what happens when an infected body is completely worn out, that means that there is no hope of it being re-used, it is expired.

The differential equations corresponding to this new model are:

$$\frac{dS}{dt}=\Pi-\beta SZ-\delta S$$
$$\frac{dZ}{dt}=\beta SZ+\zeta R-\alpha SZ-\Omega Z$$
$$\frac{dR}{dt}=\delta S+\alpha SZ-\zeta R$$

Equilibrium point analysis: Is coexistence possible?

We begin by looking for equilibrium points and then analyzing their stability. Setting the above equations to 0 and solving for (S,Z,R) we have:

$$S_0=\frac{\Pi\Omega}{\beta\Pi+\delta\Omega}$$
$$Z_0=\frac{\Pi}{\Omega}$$
$$R_0=\frac{\alpha\Pi^2+\delta\Pi\Omega}{\beta\zeta\Pi+\delta\zeta\Omega}$$

Using values from [1] for the parameters, α=.005, β=.0095, δ=.0001 and ζ=.0001. Also, from numerical experiments, the values Π=.6 and Ω=.1 were used. Substituting these values for the equilibrium point, we have (S_0,Z_0,R_0)=(10.5079, 6, 3162.87), this is the equilibrium point we will be analyzing.

We proceed by computing the Jacobian of this system about the point (S_0,Z_0,R_0).

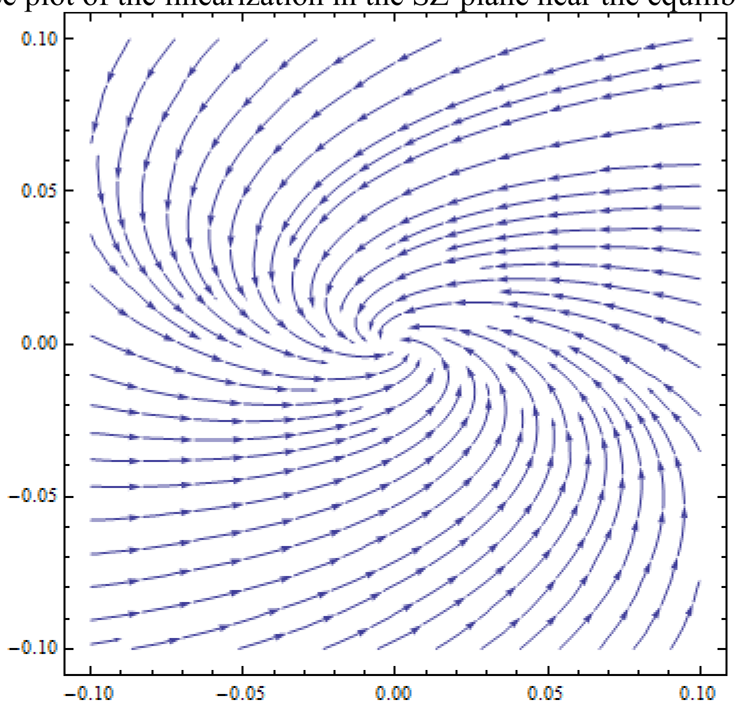
$$\begin{pmatrix} -\frac{\beta\Pi+\delta\Omega}{\Omega} & -\frac{\beta\Pi\Omega}{\beta\Pi+\delta\Omega} & 0 \\ \frac{(-\alpha+\beta)\Pi}{\Omega} & -\frac{\Omega(\alpha\Pi+\delta\Omega)}{\beta\Pi+\delta\Omega} & \zeta \\ \delta+\frac{\alpha\Pi}{\Omega} & \frac{\alpha\Pi\Omega}{\beta\Pi+\delta\Omega} & -\zeta \end{pmatrix}$$

When the values of all the parameters are substituted in, the following matrix results:

$$\begin{pmatrix} -0.0571 & -0.0998249 & 0 \\ 0.027 & -0.0527145 & 0.0001 \\ 0.0301 & 0.0525394 & -0.0001 \end{pmatrix}$$

The Eigenvalues of this matrix are -0.0549072 $\pm$ 0.0518189i, and -0.000100175, all three real parts are negative, so this equilibrium point is stable! This means that initial conditions near (10.5079, 6, 3162.87) in phase space will tend toward equilibrium!

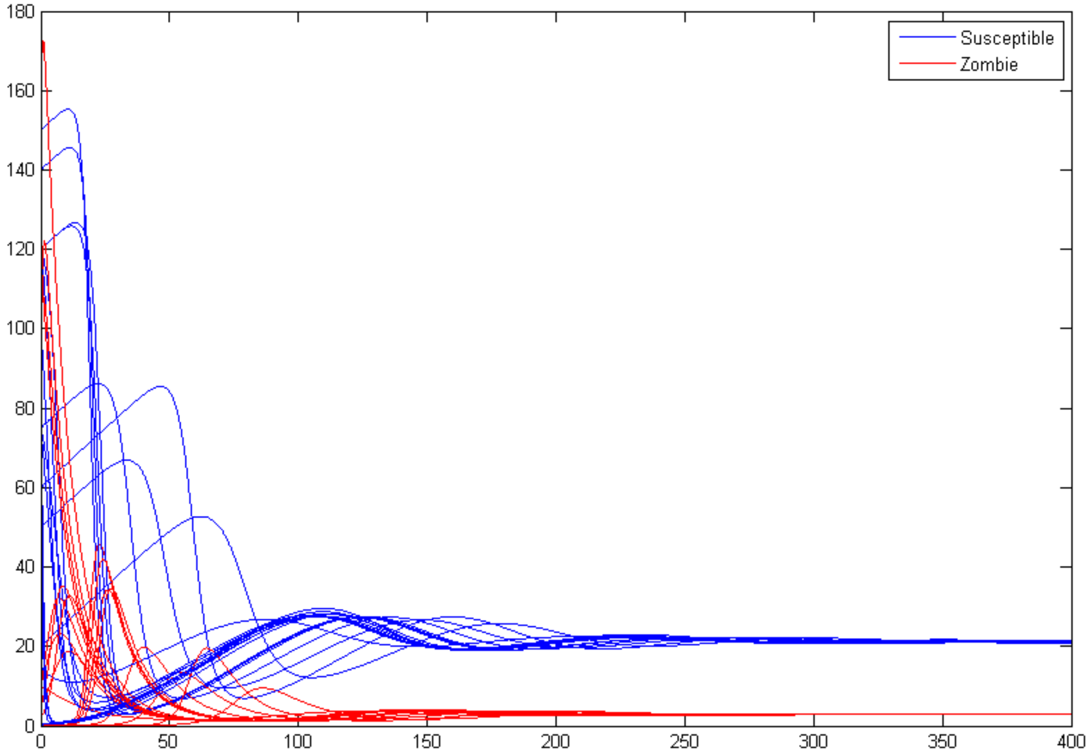
Observe the phase space plot of the linearization in the SZ-plane near the equilibrium point:



The above phase space plot was generated in Wolfram Mathematica 8. The horizontal axis is the susceptible value, and the vertical is the zombie value, the entire plot is centered about the origin because of the linearization technique used. The origin represents the point (10.5079,6,3162.87), but the R axis is ignored.

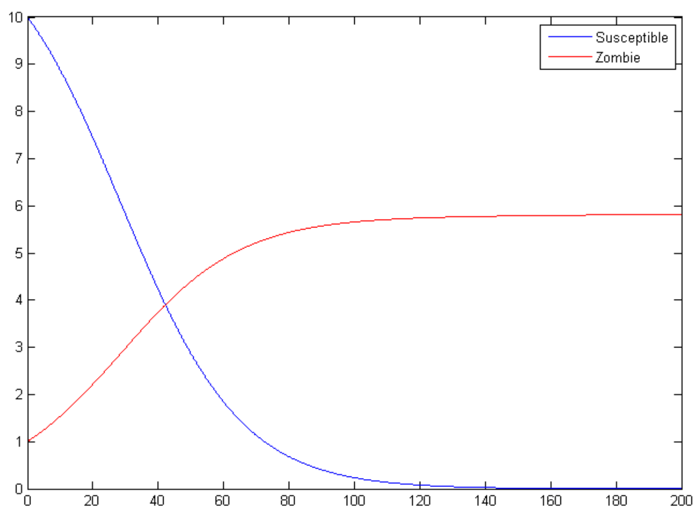
Numerical Simulations

We implemented this system of ODEs using Euler's method in MATLAB, many initial conditions were explored, all with the same long term results:

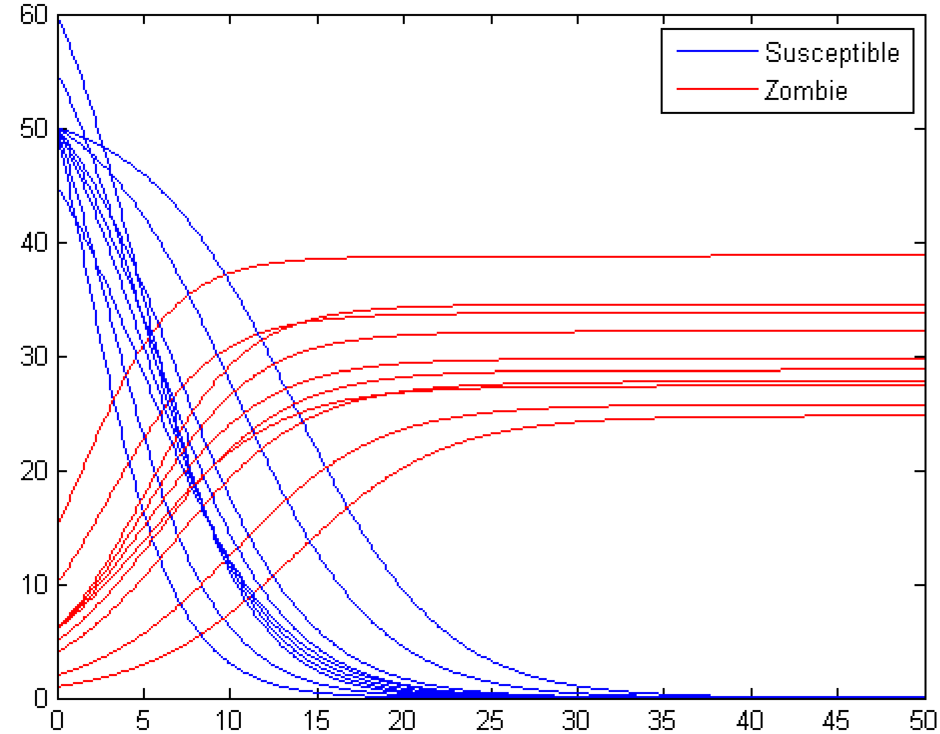


Of all of the different initial conditions, the long term values of about 3 zombies and 21 susceptibles are always found, this was suggested by the negative real parts of the eigenvalues. The surface (S,Z,R)=(3, 21,R) appears to be an attractor, this is incredibly good news! Given that zombies expire, an outbreak of them may not be a doomsday after all!

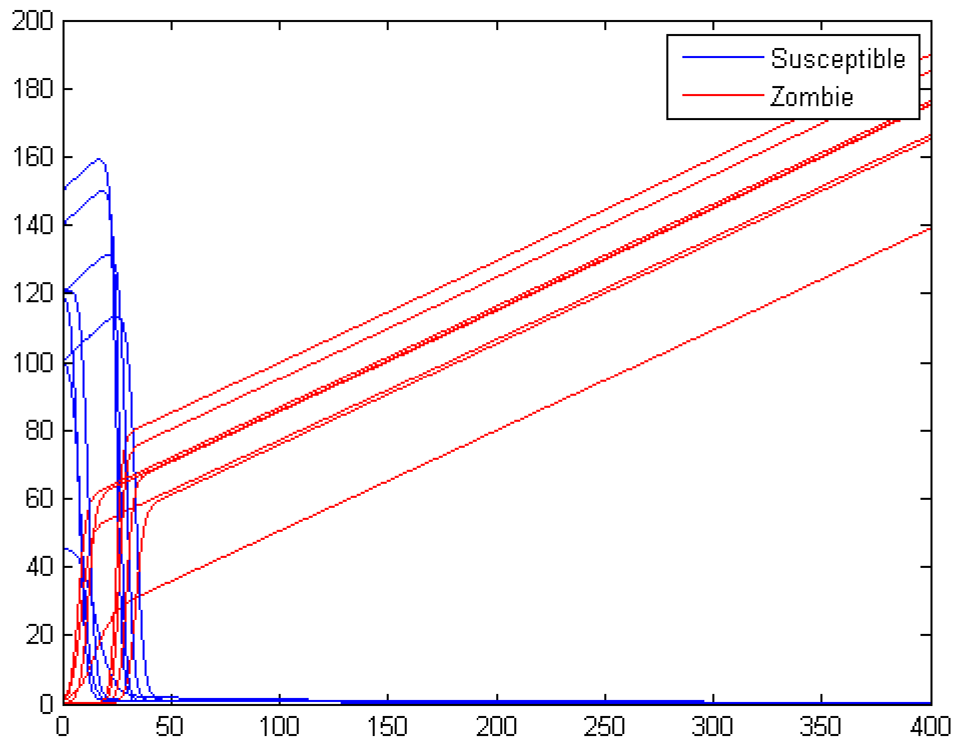
To see that this model reduces to the basic one proposed in [1], we set Π=δ=Ω=0 and run the simulation for 200 time steps:



Varying the initial conditions, we see that the long term behavior appears the same, zombies win, humans lose:



If we set $\Pi>0$, $\delta>0$ but keep $\Omega=0$, we recover the basic model, but now we explore more long term behavior, now, since the birth and death rates are positive, the zombie population grows without bound and the humans dwindle again to 0:



Conclusion

Since the main difference between the basic model [1] and the modification is the Ω term, this modification is solely responsible for the existence of the stable equilibrium that enables human-zombie coexistence. For the sake of humanity, let us hope that if a zombie outbreak were to occur, they would expire in a manner similar to that depicted in “28 Days Later”.

(MATLAB code included in Lehman_final_project.m)

References

[1] Philip Munz, Ioan Hudea, Joe Imad, and Robert J. Smith. When zombies attack!: Mathematical modellings of an outbreak of zombie infection. In Infectious Disease Modelling Research Progress, pages 133-150. Nova Science, 2009.